Covering spaces revisited

Let $p:(E,e_0)\to(B,b_0)$ be a covering map. How do the fundamental groups $\pi_1(E)$ and $\pi_1(B)$ compare to each other?

First of all, if $\tilde{h}$ is a loop at e_0 and $p_*([\tilde{h}])$ is the identity, then there's a path homotopy from H $p\circ h$ to the constant loop, so the lifting $\tilde{H}$ is a path homotopy from $\tilde{h}$ to the constant loop, so $[\tilde{h}]$ is the identity. This proves:

Thm: p_* is an injective homomorphism.

Thus, each covering p of B gives us some subgroup H of $\pi_1(B)$ that is isomorphic to $\pi_1(E)$ via p_*.

Thm: If $[f]\in p_*(\pi_1(E,e_0))$, then f lifts to a loop in E.

Pf: $\exists$ a loop $\tilde{g}$ at e_0 s.t. $p\circ\tilde{g}\simeq_p f$. But then the lifts are path homotopic so $\tilde{f}$ is a loop at e_0 as well. $\square$

It turns out:

1.) The subgroup H determines (up to "equivalence") the covering p.

2.) As long as B is path connected and "suitably nice", for each subgroup H of $\pi_1(B)$, there is some covering p whose corresponding subgroup is H.

We will talk about 1.) in this section. For 2.), you'll need to take another algebraic topology course (or read the rest of the chapter in Munkres ☺)

Equivalence of covering spaces

Def: Let $p: E \to B$ and $p': E' \to B$ be coverings. p and p' are <u>equivalent</u> if $\exists$ a homeomorphism $h: E \to E'$ s.t. $p = p' \circ h$.

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[dr, "p"'] & E' \arrow[d, "p'"] \\ & B \end{tikzcd}$$

h is an <u>equivalence of coverings</u>.

Ex: Let p and p' be coverings from $\mathbb{R}$ to S^1 defined

$$p(x) = (\cos x, \sin x), \quad p'(x) = (\cos(-x), \sin(-x)).$$

Then p and p' are equivalent via the homeomorphism $h(x) = -x$.

We want to show that if two coverings have the same corresponding subgroup of $\pi_1(B)$, then they are equivalent. First we need a definition and a more general lifting lemma:

Def: A space X is locally path connected if $\forall\, x \in X$ and every neighborhood U of x, there is a path-connected neighborhood of x $V \subseteq U$.

From now on, if $p: E \to B$ is a covering, we'll assume E and B are path connected and locally path connected.

General lifting lemma: Let $p: E \to B$ be a covering map. Let $p(e_0) = b_0$. Let Y be path connected and locally path connected and $f: Y \to B$ continuous s.t. $f(y_0) = b_0$. f can be lifted to $\tilde{f}: Y \to E$ w/ $\tilde{f}(y_0) = e_0$ iff

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0)).$$

If the lifting exists, it's unique.

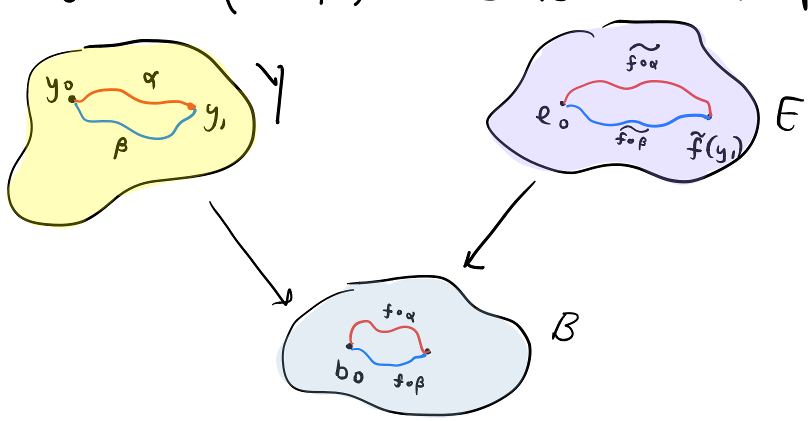
Pf: If f can be lifted, then $f = p \circ \tilde{f}$, so

$$f_*(\pi_1(Y, y_0)) = p_*(\tilde{f}_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0)).$$

For the converse, let $y_1 \in Y$. Choose a path α from y_0 to y_1.
Lift $f \circ \alpha$ to a path $\widetilde{f \circ \alpha}$ in E starting at e_0.
Define $\tilde{f}(y_1)$ to be the endpoint of this path, i.e. $\widetilde{f \circ \alpha}(1)$.

This gives a function $\tilde{f}: Y \to E$.

We first need to check $\tilde{f}$ is well-defined.
Let β be a different path from y_0 to y_1.
is a loop at y_0. Thus $f \circ (\alpha * \bar{\beta})$ lifts to a loop in E.
So the lifts piece together in E.



That is, $f \circ (\alpha * \bar{\beta})$
$= (f \circ \alpha) * \overline{(f \circ \beta)}$

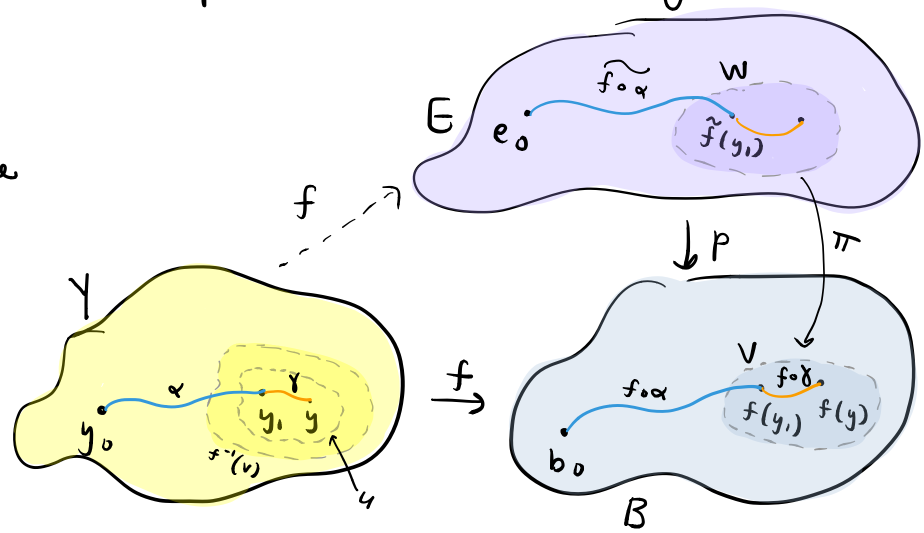
So the lift of $\overline{f \circ \beta}$ is a path from $\tilde{f}(y_1)$ to e_0 $\Rightarrow$ the lift of $f \circ \beta$ is a path from e_0 to $\tilde{f}(y_1)$.

Now we show $\tilde{f}$ is continuous. Note that we just need to show it's continuous on a neighborhood of y_1.

Let $V \subseteq B$ be an evenly covered neighborhood of $f(y_1)$. Then we can find $U \subseteq f^{-1}(V)$ a path-connected neighborhood of y_1.

Let $W \subseteq p^{-1}(V) \subseteq E$ be the slice containing $\tilde{f}(y_1)$.

Then we have a homeomorphism $\pi : W \to V$ defined $\pi = p|_W$.



For $y \in U$, we can find a path γ in U from y_1 to y, and $\pi^{-1} \circ f \circ \gamma$ is thus a lift of $f \circ \gamma$ to E starting at $\tilde{f}(y)$.

Thus, we get a lift of a path from b_0 to $f(y)$, so $\tilde{f}(y) = \pi^{-1}(f(y))$, the composition of two continuous functions, so $\tilde{f}|_U$ is continuous $\Rightarrow$ $\tilde{f}$ is continuous.

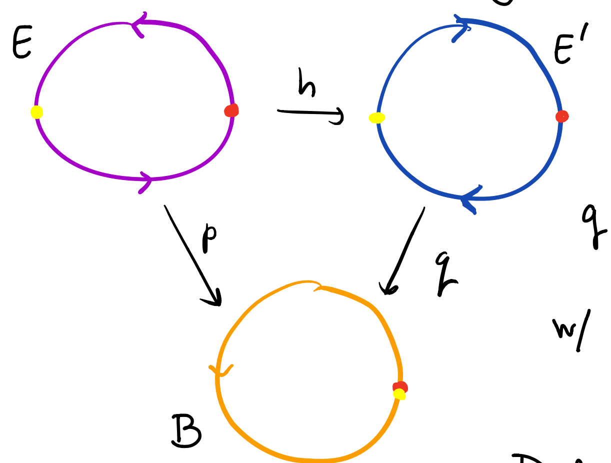
If g is another lift, then for $y_1 \in Y$, consider a path α from y_0 to y_1. Then there's a unique lifting of $f \circ \alpha$ starting at e_0, ending at $\tilde{f}(y_1)$

But $g \circ \alpha$ is a lifting of $f \circ \alpha$ starting at e_0, so

$$g(y_1) = g \circ \alpha(1) = \tilde{f}(y_1) \Rightarrow g = \tilde{f}, \text{ so the lift is unique.} \ \square$$

Now we can show equivalent coverings correspond to the same subgp of $\pi_1(B, b_0)$, as long as the maps all preserve basepoints.

Ex: Consider the 2 degree 2 coverings defined



$$p(\cos\theta, \sin\theta) = (\cos 2\theta, \sin 2\theta)$$

$$q(\cos\theta, \sin\theta) = (\cos(-2\theta), \sin(-2\theta))$$

w/ basepoint $= (1,0)$

Define $h: E \to E'$ by

$h(\cos\theta, \sin\theta) = (\cos(-\theta), \sin(-\theta))$ (or simply $(x,y) \mapsto (x,-y)$)

Clearly $q \circ h = p$, and the image of p_* and q_* corresponds to the subgroup $2\mathbb{Z}$ of $\mathbb{Z}$.

The standard covering space $p: \mathbb{R} \to S^1$ corresponds to the trivial subgroup, since $\mathbb{R}$ is simply connected.

Thm: Let $p: E \to B$ and $p': E' \to B$ be covering maps w/ $p(e_0) = p'(e_0') = b_0$. There is an equivalence $h: E \to E'$ s.t. $h(e_0) = e_0'$ $\iff$ $H = p_*(\pi_1(E, e_0))$ and $H' = p'_*(\pi_1(E', e_0'))$ are equal.

If such an h exists, it's unique.

Pf: $\Rightarrow$: If $h: E \to E'$ is such an equivalence, then $h_*(\pi_1(E,e_0)) = \pi_1(E', e_0')$. So the conclusion follows from the fact that $p'_* \circ h_* = p_*$.

$\Leftarrow$: Now assume $H = H'$. Then by the previous lemma, we get unique basepoint preserving lifts

$$\begin{tikzcd} & E' \arrow[d, "p'"] \\ E \arrow[ur, dashed, "h"] \arrow[r, "p"'] & B \end{tikzcd} , \quad \begin{tikzcd} & E \arrow[d, "p"] \\ E' \arrow[ur, dashed, "h'"] \arrow[r, "p'"'] & B \end{tikzcd}$$

so $p' \circ h = p$ and $p \circ h' = p'$

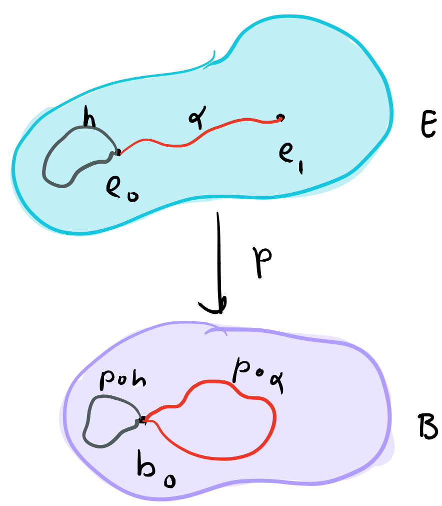
$\Longrightarrow$ $p' \circ h \circ h' = p \circ h' = p'$ and $p \circ h' \circ h = p$.

Thus

$$\begin{tikzcd} & E \arrow[d, "p"] \\ E \arrow[ur, dashed, "h' \circ h"] \arrow[r, "p"'] & B \end{tikzcd}$$

$h' \circ h$ is a lifting. So is id_E. But lifting is unique! So $h' \circ h = id_E$, and similarly $h \circ h' = id_{E'}$, so h is a homeomorphism satisfying $p = p' \circ h$, so it's an equivalence. □

If we have an equivalance $h: E \to E'$ that we don't require to preserve base points, then we'll see that the corresponding subgroups are conjugate. First let's see how changing the base point of a covering space changes the corresponding subgroup:



If $e_0, e_1 \in p^{-1}(b_0)$, and α a path from e_0 to e_1,

Let h be a loop at e_0. Then $[\alpha]^{-1} * [h] * [\alpha] \in \pi_1(E, e_1)$

If H_0 and H_1 are the subgps corresponding to e_0 and e_1, then since $p \circ \alpha$ is a loop in B, we have $[\alpha]^{-1} H_0 [\alpha] \subseteq H_1$

and similarly $[\alpha] H_1 [\alpha]^{-1} \subseteq H_0$, so the inclusions must be equalities.

Conversely, if $H_2 \subseteq \pi_1(B, b_0)$ is some other subgp conjugate to H_0, (i.e. $[\beta]^{-1} H_0 [\beta] = H_1$) then $\exists$ $e_2 = \tilde{\beta}(1)$ s.t. $H_2 = p_*(\pi_1(E, e_2))$.

This gives us the following more general statement:

Theorem: Let $p: E \to B$ and $p': E' \to B$ be covering maps. Let $p(e_0) = p'(e_0') = b_0$.

p and p' are equivalent $\iff$ $H = p_*(\pi_1(E, e_0))$ and $H' = p'_*(\pi_1(E', e_0'))$ are conjugate.

Pf: If $h: E \to E'$ is an equivalence, w/ $e_1' = h(e_0)$, let $G = p_*(\pi_*(E', e_1'))$. Then $H = G$ (by previous thm), and G is conjugate to H', by above discussion.

If H and H' are conjugate, there's some $e_1' \in E'$ s.t. $H = p'_*(\pi_*(E', e_1'))$. Thus, by the previous thm, $\exists$ an equivalence $h: E \to E'$ w/ $h(e_0) = e_1'$. $\square$

Universal covering spaces

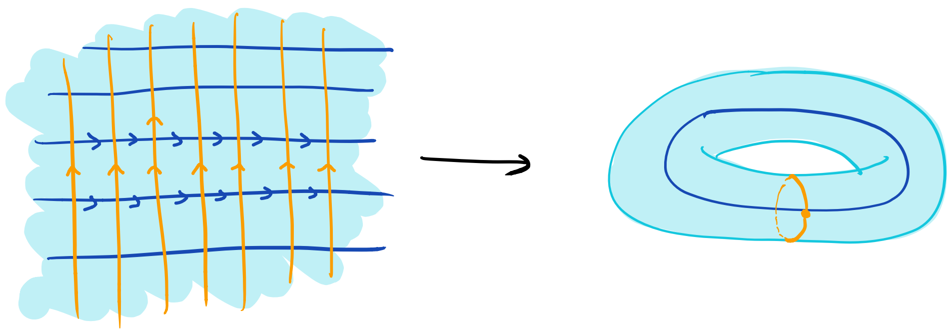
Def: If $p: E \to B$ is a covering and E is simply connected, E is called a <u>universal covering space</u> of B.

Note: If E and E' are universal covering spaces of B, then their corresponding subgroups are both trivial, so there is an equivalence $h: E \to E'$. In particular, E and E' are homeomorphic.

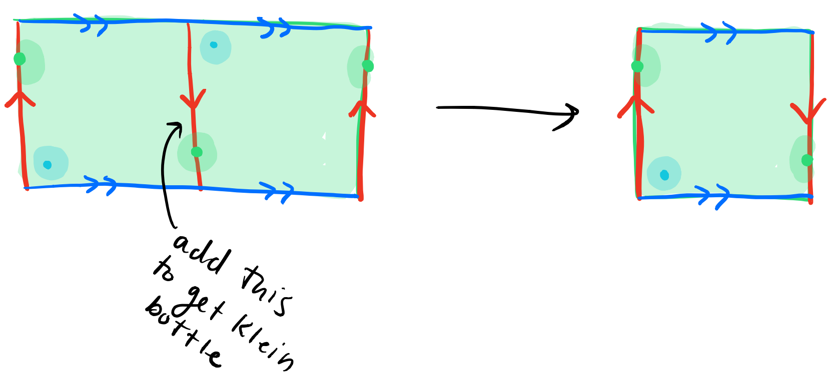
Examples

1.) $\mathbb{R}$ is the universal cover for S^1

2.) We saw a couple weeks ago that $\mathbb{R}^2$ covers the torus $S^1 \times S^1$:



3.) A Klein bottle can be covered by both the torus and $\mathbb{R}^2$. We can see the 2-to-1 covering by the torus as shown:



4.) The universal covering of the figure-eight space is the infinite tree: